Distance shifts

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Distance shifts

Or Slide shifts, or ...

- Concept comes from Aanderaa-Lewis 1974/Lewis 1979
- The article is concerned with logic and $\exists\forall\exists$ formulas
- Do not address symbolic dynamics at all
- Proofs here are only inspired by it, but uses more formal language theory and less arithmetic.

For more details, read our book chapter on Pascal's webpage.

Plan

1. Definitions
2. 2D
3. Proofs

One dimensional shifts

- A a finite alphabet
- A (sub)shift is a topologically closed, translation invariant subset of $A^{\mathbb{Z}}$
- Equivalently, there exists a set F s.t. S is exactly the set of biinfinite words that do not contain factors in F.
- S is an SFT if we can chose F finite
- S is sofic if we can chose F a regular language

Similar definitions in dimension 2.

Distance/Slide shifts

Definition

Let L be a SFT/sofic shift of $(A \times B)^{\mathbb{Z}}$.
The slide shift L^{Δ} associated with L is

$$\left\{(x, y) \in (A \times B)^{\mathbb{Z}} \middle| \forall i, (x, \sigma^i(y)) \in L\right\}$$

where σ is the translation

A slide shift is of course a subshift.

Example

Let $A = B = \{0,1\}$. We see x as being "on top" of y.
Let L be the sofic shift which forbids:

$$\begin{array}{ccc|ccc|ccc|ccc} 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 0 & 0 & \cdots & 1 \\ 1 & \cdots & 0 & 0 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \end{array}$$

In other words:

If we have $\begin{smallmatrix}1\\1\end{smallmatrix}$, we may have only $\begin{smallmatrix}1\\1\end{smallmatrix}$ and $\begin{smallmatrix}0\\0\end{smallmatrix}$.

What is L^{Δ} ?

- If x contains 10^n1 and y contains 10^m1, then we must have $n = m$ (True also for $n = 0, n = \infty$)

x	...	1	0	0	0	1	...	...	...
y	...	1	0	0	0	0	0	1	...

Therefore L^Δ consists in three parts:

- $x = 0^{\mathbb{Z}}$, $y \in \{0, 1\}^{\mathbb{Z}}$
- $x \in \{0, 1\}^{\mathbb{Z}}$, $y = 0^{\mathbb{Z}}$
- $x = (10^n)^{\mathbb{Z}}$ and $y = (10^n)^{\mathbb{Z}}$ upto shift for some n (possibly $n = \infty$)

- If x contains 10^n1 and y contains 10^m1, then we must have $n = m$ (True also for $n = 0, n = \infty$)

x	...	1	0	0	0	1	...	...	...
$\sigma(y)$	...	...	1	0	0	0	1	...	...

Therefore L^Δ consists in three parts:

- $x = 0^{\mathbb{Z}}$, $y \in \{0,1\}^{\mathbb{Z}}$
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- $x = (10^n)^{\mathbb{Z}}$ and $y = (10^n)^{\mathbb{Z}}$ upto shift for some n (possibly $n = \infty$)

Theorems

Theorem

There exists an aperiodic slide shift.

Theorem

There is no algorithm to decide if a slide shift is empty

This talk

- Why we care
- How to prove it

Plan

2D vs 1D

$\mathbb{Z}^2$ acts naturally on slide shifts.

Any slide shift is conjugated to a 2D sofic shift.

x_{-4} y_5	x_{-3} y_5	x_{-2} y_5	x_{-1} y_5	x_0 y_5	x_1 y_5	x_2 y_5	x_3 y_5	x_4 y_5
x_{-4} y_4	x_{-3} y_4	x_{-2} y_4	x_{-1} y_4	x_0 y_4	x_1 y_4	x_2 y_4	x_3 y_4	x_4 y_4
x_{-4} y_3	x_{-3} y_3	x_{-2} y_3	x_{-1} y_3	x_0 y_3	x_1 y_3	x_2 y_3	x_3 y_3	x_4 y_3
x_{-4} y_2	x_{-3} y_2	x_{-2} y_2	x_{-1} y_2	x_0 y_2	x_1 y_2	x_2 y_2	x_3 y_2	x_4 y_2
x_{-4} y_1	x_{-3} y_1	x_{-2} y_1	x_{-1} y_1	x_0 y_1	x_1 y_1	x_2 y_1	x_3 y_1	x_4 y_1
x_{-4} y_0	x_{-3} y_0	x_{-2} y_0	x_{-1} y_0	x_0 y_0	x_1 y_0	x_2 y_0	x_3 y_0	x_4 y_0
x_{-4} y_{-1}	x_{-3} y_{-1}	x_{-2} y_{-1}	x_{-1} y_{-1}	x_0 y_{-1}	x_1 y_{-1}	x_2 y_{-1}	x_3 y_{-1}	x_4 y_{-1}
x_{-4} y_{-2}	x_{-3} y_{-2}	x_{-2} y_{-2}	x_{-1} y_{-2}	x_0 y_{-2}	x_1 y_{-2}	x_2 y_{-2}	x_3 y_{-2}	x_4 y_{-2}
x_{-4}	x_{-3}	x_{-2}	x_{-1}	x_0	x_1	x_2	x_3	x_4

Corollaries

There exists an aperiodic 2D-SFT.

The Domino Problem is undecidable for 2D-SFTs.

(Domino Problem: decide whether a SFT is empty).

Determinism

Theorem

Every 1D sofic shift has a right resolving SFT cover (= determinisation of the finite automaton).

Applying this to $L \subseteq (A \times B)^{\mathbb{Z}}$, we get a (nearest neighbour) SFT $N \subseteq (A \times B \times C)^{\mathbb{Z}}$ s.t.

$$\begin{array}{|c|c|c|c|c|c|c|} \hline x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 \\ \hline \end{array} \in L$$

iff there exists (z_i) s.t.

$$\begin{array}{|cc|cc|cc|cc|cc|cc|cc|} \hline x_1 & & x_2 & & x_3 & & x_4 & & x_5 & & x_6 & & x_7 & \\ & z_1 & & z_2 & & z_3 & & z_4 & & z_5 & & z_6 & & z_7 \\ y_1 & & y_2 & & y_3 & & y_4 & & y_5 & & y_6 & & y_7 & \\ \hline \end{array} \in N$$

and z_{i+1} is a function of x_i, y_i, z_i.

x_{-2} y_4 $z_{-2,4}$	x_{-1} y_4 $z_{-1,4}$	x_0 y_4 $z_{0,4}$	x_1 y_4 $z_{1,4}$	x_2 y_4 $z_{2,4}$	x_3 y_4 $z_{3,4}$
x_{-2} y_3 $z_{-2,3}$	x_{-1} y_3 $z_{-1,3}$	x_0 y_3 $z_{0,3}$	x_1 y_3 $z_{1,3}$	x_2 y_3 $z_{2,3}$	x_3 y_3 $z_{3,3}$
x_{-2} y_2 $z_{-2,2}$	x_{-1} y_2 $z_{-1,2}$	x_0 y_2 $z_{0,2}$	x_1 y_2 $z_{1,2}$	x_2 y_2 $z_{2,2}$	x_3 y_2 $z_{3,2}$
x_{-2} y_1 $z_{-2,1}$	x_{-1} y_1 $z_{-1,1}$	x_0 y_1 $z_{0,1}$	x_1 y_1 $z_{1,1}$	x_2 y_1 $z_{2,1}$	x_3 y_1 $z_{3,1}$
x_{-2} y_0 $z_{-2,0}$	x_{-1} y_0 $z_{-1,0}$	x_0 y_0 $z_{0,0}$	x_1 y_0 $z_{1,0}$	x_2 y_0 $z_{2,0}$	x_3 y_0 $z_{3,0}$
x_{-2} y_{-1} $z_{-2,-1}$	x_{-1} y_{-1} $z_{-1,-1}$	x_0 y_{-1} $z_{0,-1}$	x_1 y_{-1} $z_{1,-1}$	x_2 y_{-1} $z_{2,-1}$	x_3 y_{-1} $z_{3,-}$
x_{-2} y_{-2} $z_{-2,-2}$	x_{-1} y_{-2} $z_{-1,-2}$	x_0 y_{-2} $z_{0,-2}$	x_1 y_{-2} $z_{1,-2}$	x_2 y_{-2} $z_{2,-2}$	x_3 y_{-2} $z_{3,-}$
x_{-2} y_{-3} $z_{-2,-3}$	x_{-1} y_{-3} $z_{-1,-3}$	x_0 y_{-3} $z_{0,-3}$	x_1 y_{-3} $z_{1,-3}$	x_2 y_{-3} $z_{2,-3}$	x_3 y_{-3} $z_{3,-}$

x_{-2} y_4 $z_{-2,4}$	x_{-1} y_4 $z_{-1,4}$	x_0 y_4 $z_{0,4}$	x_1 y_4 $z_{1,4}$	x_2 y_4 $z_{2,4}$	x_3 y_4 $z_{3,4}$
x_{-2} y_3 $z_{-2,3}$	x_{-1} y_3 $z_{-1,3}$	x_0 y_3 $z_{0,3}$	x_1 y_3 $z_{1,3}$	x_2 y_3 $z_{2,3}$	x_3 y_3 $z_{3,3}$
x_{-2} y_2 $z_{-2,2}$	x_{-1} y_2 $z_{-1,2}$	x_0 y_2 $z_{0,2}$	x_1 y_2 $z_{1,2}$	x_2 y_2 $z_{2,2}$	x_3 y_2 $z_{3,2}$
x_{-2} y_1 $z_{-2,1}$	x_{-1} y_1 $z_{-1,1}$	x_0 y_1 $z_{0,1}$	x_1 y_1 $z_{1,1}$	x_2 y_1 $z_{2,1}$	x_3 y_1 $z_{3,1}$
x_{-2} y_0 $z_{-2,0}$	x_{-1} y_0 $z_{-1,0}$	x_0 y_0 $z_{0,0}$	x_1 y_0 $z_{1,0}$	x_2 y_0 $z_{2,0}$	x_3 y_0 $z_{3,0}$
x_{-2} y_{-1} $z_{-2,-1}$	x_{-1} y_{-1} $z_{-1,-1}$	x_0 y_{-1} $z_{0,-1}$	x_1 y_{-1} $z_{1,-1}$	x_2 y_{-1} $z_{2,-1}$	x_3 y_{-1} $z_{3,-}$
x_{-2} y_{-2} $z_{-2,-2}$	x_{-1} y_{-2} $z_{-1,-2}$	x_0 y_{-2} $z_{0,-2}$	x_1 y_{-2} $z_{1,-2}$	x_2 y_{-2} $z_{2,-2}$	x_3 y_{-2} $z_{3,-}$
x_{-2} y_{-3} $z_{-2,-3}$	x_{-1} y_{-3} $z_{-1,-3}$	x_0 y_{-3} $z_{0,-3}$	x_1 y_{-3} $z_{1,-3}$	x_2 y_{-3} $z_{2,-3}$	x_3 y_{-3} $z_{3,-}$

Corollaries

There exists an aperiodic 2D-SFT with one direction of expansiveness.

The Domino Problem is undecidable for 2D-SFTs with directions of expansiveness.

There exists a cellular automaton that is nilpotent on periodic configurations but not nilpotent.

Nilpotency is undecidable for cellular automata.

Last result was proven by Kari (1990), but is actually already contained explicitly in Aanderaa-Lewis (1974).

Plan

Theorems

Theorem

There exists an aperiodic slide shift.

Theorem

There is no algorithm to decide if a slide shift is empty

What can we do with slide shifts ?

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Not clear, but here is an idea:

- Let's try to do $L^{\Delta} = S \times S$ for some subshift S.
- Find good subshifts S for which comparing two elements $x, y \in S$ gives us many information about S
- Can we code S by only specifying how elements of S differ from each other?

Toeplitz subshift

Definition

Let $p > 0$.
If x is a integer, let

- $a(x)$ be the last nonzero digit in the writing of x in base p (a is multivalued for $x = 0$).
- $v(x)$ (valuation/level) is the position of this digit ($= \infty$ for $x = 0$)

For $p = 5$:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	..
$a(n)$	1	2	3	4	1	1	2	3	4	2	1	2	3	4	3	..

Let $u_n = a(n)$ and S_p be the subshift generated by u.

Alternate definition

Definition

A p-adic integer is an "integer" with an infinite expansion in base p.
Let $\mathbb{Z}_p$ be the set of p-adic integers.
$a(x)$ can be defined more generally on $\mathbb{Z}_p$.

Then

$$S_p = \{(n \to a(x+n)) | x \in \mathbb{Z}_p\}$$

Theorem

Theorem

For $p > 6$, $S_p \times S_p$ is a slide shift

Idea of the proof:

Let $x, y \in S_p$.
If we only look at x and y only at the positions where they differ, we almost recover an element of S_p

What I mean

$u(n) = a(n)$
$v(n) = a(n + 10)$

u 12341123421234312344123411234112342123431234412342123411234212343123441234312341123421234312344 1

v 12343123441234112341123421234312344123421234112342123431234412343123411234212343123441234412341 1

u_n if $u_n \neq v_n$ ____1____2____3____4____1____1____2____3____4_________1____2_________4____3____1____2____3____4_

What I mean

$u(n) = a(n)$
$v(n) = a(n + 10)$

u	123411234212343123441234112341123421234312344123421234112342123431234412343123411234212343123441
v	123431234412341123411234212343123441234212341123421234312344123431234112342123431234412344123411
u_n if $u_n \neq v_n$	____1____2____3____4____1____1____2____3____4_________1____2_________4____3____1____2____3____4_

1234112341243123441234112341123412431234412341234112341243

Main idea

Let $q = p - 1$.

Proposition

Let $x, y \in S_p$ that differ on more than one value. Then x restricted to the position where it differs from y, is formed of concatenations of blocks of the form:

$$1 \cdot 2 \cdot 3 \cdots q \cdot 1 \cdot 1 \cdot 2 \cdot 3 \cdots q \cdot 2 \cdots\cdots\cdots\cdots\cdots q \cdot q \cdot 1 \cdot 2 \cdot 3 \cdots q \cdot ?$$

with at most 4 symbols missing per block.

Main idea

Theorem

Let x, y be two words. Suppose that:

- *x and y are infinite concatenations of blocks with no symbols missing*
- *For all i, the word x restricted to the positions where it differs from $\sigma^i(y)$ is of the form of the previous proposition*

Then $x, y \in S_p$.

(That's the ugly part of the proof)

Corollary

$S_p \times S_p$ is a slide shift.

How to prove undecidability of emptyness for slide shifts ?

One can code 1D one-sided SFTs with condition on the origin inside slide shifts

One can code 2D SFTs on a quarter plane with condition on the origin inside slide shifts

Emptyness of such SFTs is trivially undecidable.

Idea

- Suppose our SFT has two colors : red , blue.
- red cannot appear after blue
- The origin is red.

Idea of the proof:

- Each symbol will now have a color (red/blue)
- Positions with valuation(level) i will encode the color of the i-th letter of the SFT.

Idea

This is realized by the following slide shift: $(x, y) \in L^{\Delta}$ if:

- x and y are infinite concatenations of

$$1 \cdot 2 \cdot 3 \cdots q \cdot 1 \cdot 1 \cdot 2 \cdot 3 \cdots q \cdot 2 \cdots\cdots\cdots\cdots q \cdot q \cdot 1 \cdot 2 \cdot 3 \cdots q \cdot ?$$

- For all i, the word x restricted to the positions where it differs from $\sigma^i(y)$ is composed of concatenations of words of the form

$$\underline{1 \cdot 2 \cdot 3 \cdots q \cdot} \boxed{1} \cdot \underline{1 \cdot 2 \cdot 3 \cdots q \cdot} \boxed{2} \underline{\cdots\cdots\cdots\cdots q \cdot} \boxed{q} \underline{\cdot 1 \cdot 2 \cdot 3 \cdots q} \cdot ?$$

  with at most 4 symbols missing per block, where:
  - All the underlined symbols have the same color
  - The boxed symbols have the same color
  - If the underlined symbols are blue, then the box symbols are blue.

End of the idea

To obtain the undecidability:

- Let X be a 2D SFT on a quarter of the plane with initial condition
- Using a slide shift on $S_p \times S_q$ for p prime with q, we can encode X in a slide shift.
- The slide shift will be empty iff X is empty

Open questions

- Find what can and cannot be done by slide shifts.
- Simplify the proofs