Distance shifts

E. Jeandel (joint work with P. Vanier)

Loria (Nancy)
Distance shifts

Or Slide shifts, or ...

- Concept comes from Aanderaa-Lewis 1974/Lewis 1979
- The article is concerned with logic and $\exists\forall\exists$ formulas
- Do not address symbolic dynamics at all
- Proofs here are only inspired by it, but uses more formal language theory and less arithmetic.

For more details, read our book chapter on Pascal’s webpage.
Plan

1. Definitions
2. 2D
3. Proofs
One dimensional shifts

- $A$ a finite alphabet
- A (sub)shift is a topologically closed, translation invariant subset of $A^\mathbb{Z}$
- Equivalently, there exists a set $F$ s.t. $S$ is exactly the set of biinfinite words that do not contain factors in $F$.
- $S$ is an SFT if we can chose $F$ finite
- $S$ is sofic if we can chose $F$ a regular language

Similar definitions in dimension 2.
Distance/Slide shifts

Definition

Let $L$ be a SFT/sofic shift of $(A \times B)^\mathbb{Z}$. The slide shift $L^\Delta$ associated with $L$ is

$$\left\{(x, y) \in (A \times B)^\mathbb{Z} \mid \forall i, (x, \sigma^i(y)) \in L \right\}$$

where $\sigma$ is the translation

A slide shift is of course a subshift.
Example

Let \( A = B = \{0, 1\} \). We see \( x \) as being “on top” of \( y \).
Let \( L \) be the sofic shift which forbids:

\[
\begin{array}{c|c|c|c|c|c|c|}
1 & \cdots & 1 & 1 & \cdots & 1 & 1 \\
1 & \cdots & 0 & 0 & \cdots & 1 & 1 \\
\end{array}
\]

In other words:

If we have \( \begin{array}{c} 1 \\ 1 \end{array} \), we may have only \( \begin{array}{c} 1 \\ 1 \end{array} \) and \( \begin{array}{c} 0 \\ 0 \end{array} \).

What is \( L^\Delta \) ?
If $x$ contains $10^n1$ and $y$ contains $10^m1$, then we must have $n = m$ (True also for $n = 0, n = \infty$)

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Therefore $L^\Delta$ consists in three parts:

1. $x = 0^\mathbb{Z}$, $y \in \{0, 1\}^\mathbb{Z}$
2. $x \in \{0, 1\}^\mathbb{Z}$, $y = 0^\mathbb{Z}$
3. $x = (10^n)^\mathbb{Z}$ and $y = (10^n)^\mathbb{Z}$ upto shift for some $n$ (possibly $n = \infty$)
• If $x$ contains $10^n1$ and $y$ contains $10^m1$, then we must have $n = m$
(True also for $n = 0, n = \infty$)

| $x$     | ... | 1  | 0 | 0 | 0 | 1 | ... | ... | ...
|---------|-----|----|---|---|---|---|-----|-----|-----
| $\sigma(y)$ | ... | ... | 1 | 0 | 0 | 0 | 1 | ... | ...

Therefore $L^\Delta$ consists in three parts:

• $x = 0^\mathbb{Z}, y \in \{0, 1\}^\mathbb{Z}$
• $x \in \{0, 1\}^\mathbb{Z}, y = 0^\mathbb{Z}$
• $x = (10^n)^\mathbb{Z}$ and $y = (10^n)^\mathbb{Z}$ upto shift for some $n$ (possibly $n = \infty$)
Theorems

Theorem

There exists an aperiodic slide shift.

Theorem

There is no algorithm to decide if a slide shift is empty.
This talk

- Why we care
- How to prove it
$\mathbb{Z}^2$ acts naturally on slide shifts.

Any slide shift is conjugated to a 2D sofic shift.
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Corollaries

There exists an aperiodic 2D-SFT.

The Domino Problem is undecidable for 2D-SFTs.

(Domino Problem: decide whether a SFT is empty).
Determinism

**Theorem**

*Every 1D sofic shift has a right resolving SFT cover (= determinisation of the finite automaton).*

Applying this to \( L \subseteq (A \times B)^\mathbb{Z} \), we get a (nearest neighbour) SFT \( N \subseteq (A \times B \times C)^\mathbb{Z} \) s.t.

\[
\begin{array}{cccccccc}
  x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\
  y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7
\end{array} \in L
\]

iff there exists \((z_i)\) s.t.

\[
\begin{array}{cccccccc}
  x_1 & z_1 & x_2 & z_2 & x_3 & z_3 & x_4 & z_4 & x_5 & z_5 & x_6 & z_6 & x_7 & z_7 \\
  y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7
\end{array} \in N
\]

and \(z_{i+1}\) is a function of \(x_i, y_i, z_i\).
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<td>$z_{3,-3}$</td>
</tr>
<tr>
<td>$y_{-3}$</td>
<td></td>
<td>$y_{-3}$</td>
<td></td>
<td>$y_{-3}$</td>
<td></td>
<td>$y_{-3}$</td>
<td></td>
<td>$y_{-3}$</td>
<td></td>
<td>$y_{-3}$</td>
<td></td>
</tr>
</tbody>
</table>
Corollaries

There exists an aperiodic 2D-SFT with one direction of expansiveness.

The Domino Problem is undecidable for 2D-SFTs with directions of expansiveness.

There exists a cellular automaton that is nilpotent on periodic configurations but not nilpotent.

Nilpotency is undecidable for cellular automata.

Last result was proven by Kari (1990), but is actually already contained explicitly in Aanderaa-Lewis (1974).
Plan

1 Definitions

2 2D

3 Proofs
Theorems

Theorem
There exists an aperiodic slide shift.

Theorem
There is no algorithm to decide if a slide shift is empty

What can we do with slide shifts?
What can we do with slide shifts?

Not clear, but here is an idea:

- Let’s try to do $L^\Delta = S \times S$ for some subshift $S$.
- Find good subshifts $S$ for which comparing two elements $x, y \in S$ gives us many information about $S$.
- Can we code $S$ by only specifying how elements of $S$ differ from each other?
Toeplitz subshift

**Definition**

Let $p > 0$. If $x$ is an integer, let

- $a(x)$ be the last nonzero digit in the writing of $x$ in base $p$ ($a$ is multivalued for $x = 0$).
- $\nu(x)$ (valuation/level) is the position of this digit ($= \infty$ for $x = 0$)

For $p = 5$:

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>..</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a(n)$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>..</td>
</tr>
</tbody>
</table>

Let $u_n = a(n)$ and $S_p$ be the subshift generated by $u$. 
A $p$-adic integer is an "integer" with an infinite expansion in base $p$. Let $\mathbb{Z}_p$ be the set of $p$-adic integers. $a(x)$ can be defined more generally on $\mathbb{Z}_p$.

Then

$$S_p = \{(n \rightarrow a(x + n)) | x \in \mathbb{Z}_p\}$$
Theorem

For $p > 6$, $S_p \times S_p$ is a slide shift

Idea of the proof:

Let $x, y \in S_p$. If we only look at $x$ and $y$ only at the positions where they differ, we almost recover an element of $S_p$
What I mean

\[ u(n) = a(n) \]
\[ v(n) = a(n + 10) \]
What I mean

\[ u(n) = a(n) \]
\[ v(n) = a(n + 10) \]
Let $q = p - 1$.

**Proposition**

Let $x, y \in S_p$ that differ on more than one value. Then $x$ restricted to the position where it differs from $y$, is formed of concatenations of blocks of the form:

$$1 \cdot 2 \cdot 3 \cdots q \cdot 1 \cdot 1 \cdot 2 \cdot 3 \cdots q \cdot 2 \cdots \cdots \cdots \cdots q \cdot q \cdot 1 \cdot 2 \cdot 3 \cdots q \cdot ?$$

with at most 4 symbols missing per block.
Main idea

Theorem

Let $x, y$ be two words. Suppose that:

- $x$ and $y$ are infinite concatenations of blocks with no symbols missing
- For all $i$, the word $x$ restricted to the positions where it differs from $\sigma^i(y)$ is of the form of the previous proposition

Then $x, y \in S_p$.

(That’s the ugly part of the proof)
Corollary

$S_p \times S_p$ is a slide shift.
How to prove undecidability of emptiness for slide shifts?

One can code 1D one-sided SFTs with condition on the origin inside slide shifts.

One can code 2D SFTs on a quarter plane with condition on the origin inside slide shifts.

Emptiness of such SFTs is trivially undecidable.
Idea

Suppose our SFT has two colors: red, blue.
red cannot appear after blue
The origin is red.

Idea of the proof:
Each symbol will now have a color (red/blue)
Positions with valuation(level) $i$ will encode the color of the $i$-th letter of the SFT.
Idea

This is realized by the following slide shift: \((x, y) \in L^\Delta\) if:

- \(x\) and \(y\) are infinite concatenations of

\[
1 \cdot 2 \cdot 3 \cdots q \cdot 1 \cdot 1 \cdot 2 \cdot 3 \cdots q \cdot 2 \cdots \cdots \cdots q \cdot q \cdot 1 \cdot 2 \cdot 3 \cdots q \cdot ?
\]

- For all \(i\), the word \(x\) restricted to the positions where it differs from \(\sigma^i(y)\) is composed of concatenations of words of the form

\[
\underline{1 \cdot 2 \cdot 3 \cdots q} \cdot \boxed{1} \cdot 1 \cdot 2 \cdot 3 \cdots q \cdot \boxed{2} \cdots \cdots \cdots q \cdot \boxed{q} \cdot 1 \cdot 2 \cdot 3 \cdots q \cdot ?
\]

with at most 4 symbols missing per block, where:

- All the underlined symbols have the same color
- The boxed symbols have the same color
- If the underlined symbols are blue, then the box symbols are blue.
To obtain the undecidability:

- Let $X$ be a 2D SFT on a quarter of the plane with initial condition.
- Using a slide shift on $S_p \times S_q$ for $p$ prime with $q$, we can encode $X$ in a slide shift.
- The slide shift will be empty iff $X$ is empty.
Open questions

- Find what can and cannot be done by slide shifts.
- Simplify the proofs