Decidability and S-adic dynamics

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Dyadisc3: Decidability and dynamical systems

S-adic expansions

- Let $\mathcal{S}$ be a set S of substitutions on the alphabet $\mathcal{A}$
- Let $s = (\sigma_n)_{n\in\mathbb{N}} \in S^{\mathbb{N}}$ a sequence of substitutions (directive sequence)
- Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of letters in $\mathcal{A}$

We say that the infinite word $u \in \mathcal{A}^{\mathbb{N}}$ admits $(\sigma_n, a_n)_n$ as an S-adic representation if

$$u = \lim_{n\to\infty} \sigma_0\sigma_1\cdots\sigma_{n-1}(a_n)$$

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The terminology comes from Vershik adic transformations
Bratteli diagrams

S stands for substitution, adic for the inverse limit

Topological isomorphism

What changes in terms of decidability from the substitutive case to S-adic case ?

One can decide whether two minimal substitution subshifts are topologically isomorphic
and even whether one is a factor of the other [Durand-Leroy]

Substitution Sturmian subshifts have quadratic parameters

What about Sturmian subshifts? Π_1

Outline

- Subshifts and computability
- Computability for S-adic systems
- Effectiveness for Sturmian shifts (higher-dimensional case)
- Recognizability

Subshifts and computability

A subshift can be described/presented

- by its language
- by its set of forbidden factors $\rightsquigarrow$ presentation of a subshift

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Computability notions Computability of the set of forbidden words.

- Finite presentation: SFT/sofic
- Computably enumerable presentation: effectively closed

[Jeandel-Vanier] Enumeration reducibility

SFT and sofic subshifts

- Subshifts of finite type (SFT) are the subshifts defined by a finite set of forbidden patterns.
- Sofic subshifts are images of SFT under a factor map.
- A factor map$\pi : X \to Y$ between two subshifts X and Y is a continuous, surjective map such that

$$\pi \circ T = T \circ \pi,$$

  where T is the shift.
- A factor map is a sliding block code (defined by a local rule/CA) [Curtis-Hedlund-Lyndon].
- Example: add colorations. Take a larger alphabet for X: X is the SFT, Y is the sofic shift.

Computable subshifts

A hierarchy of subshifts

- SFT
- Sofic (we loose information through projection)

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Recursively enumerable: there exists a Turing machine which enumerates all elements of the language.

- Π_1^0-computable or effective: the complement of the language is recursively enumerable (recursive).
- Σ_1^0-computable: the language is recursively enumerable.
- Δ_1^0-computable or decidable or recursive: its language is recursive (decidable).

Enumeration of forbidden factors

If an effective subshift is defined by a recursively enumerable set of forbidden factors, it can also be defined by a recursive =decidable set of forbidden factors.

• Indeed, let $(u_n)_n$ be the sequence of forbidden patterns generated by the Turing machine.

• One replaces $(u_n)_n$ by a sequence of forbidden patterns $(v_n)_n$ with increasing size defining the same shift. Indeed, when one enumerates $(u_n)_n$, if one adds a pattern w whose size is smaller than the previous ones, then, instead of adding w, one adds all the patterns that contain w whose size is larger than the size of the pattern previously added to $(v_n)_n$.

• The sequence (v_n) is recursively enumerable and increasing, it is thus recursive: indeed, to know if a pattern w is the sequence $(v_n)_n$, one enumerates all the patterns until being larger than the size of w.

About the computability of frequencies

Computable frequencies: there exists an algorithm that takes as input a pattern and a precision, and that outputs an approximation of this frequency with respect to this precision

$\rightsquigarrow$ Computable pattern frequencies/shift-invariant measure

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Computability of letter frequencies does not say much on the algorithmic complexity of a subshift.

- Take a subshift $X \subset \{0,1\}^{\mathbb{Z}}$ and consider the subshift Y obtained by applying to each configuration of X the substitution

$$0 \mapsto 01, 1 \mapsto 10.$$

  The subshift Y admits letter frequencies (they are both equal to $1/2$), and it has the same algorithmic complexity as X.

Effectiveness for shifts

Fact [B.-Fernique-Sablik] Let X be a subshift.

- If X is effective and uniquely ergodic, then its invariant measure is computable and X is decidable.
- If X is minimal and its frequencies are computable, then its language is recursively enumerable.
- If X is minimal and effective, then X is decidable.

Remark Existence of frequencies implies unique ergodicity. Unique ergodicity means uniformity in the convergence.

Effectiveness for shifts

We assume X effective and uniquely ergodic. Let us prove that the frequency of any pattern is computable.

- Consider the following algorithm that takes as an argument the parameter e for the precision. We consider a finite pattern p.
- At step n, one produces all 'square' patterns of size n that do not contain the n first forbidden patterns.
- For each of these square patterns, one computes the number of occurrences of p in it, divided by $(2n+1)^d$.
- We continue until these quantities belong to an interval of length e.
- This algorithm then stops (compactness=subshift + unique ergodicity=uniform frequencies), and taking an element of the interval provides an approximation of the frequency of p up to precision e.

- It remains to prove that the algorithm stops. Suppose it does not, then, for all n, one can find two patterns of size n, x_n and x'_n, that do not contain the n first forbidden patterns and such that

$$||x_n|_p/(2n+1)^d - |x'_n|_p/(2n+1)^d| > e.$$

  By compactness, we can extract two configurations x and x' that do not contain forbidden patterns (they thus belong to the subshift X) such that the frequency of p in x is distinct from the frequency of p in x'. This contradicts the unique ergodicity of X.

We assume X minimal with computable pattern frequencies. We prove that the language is recursively enumerable.

- Frequencies are positive by minimality.
- Even if the frequencies are computable, one cannot decide whether the frequency of a given pattern is equal to zero or not, hence we cannot decide whether this pattern belongs to the language or not.
- However, one can decide whether the frequency of a pattern is larger than a given value. This thus implies that the language is recursively enumerable.

Computability for S-adic systems

Geometrical substitutions and tilings

Let $\phi : \mathbb{R}^d \to \mathbb{R}^d$ be an expanding linear map

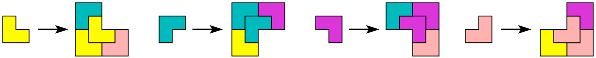
Principle One takes

- a finite number of prototiles $\{T_1, T_2, \ldots, T_m\}$
- an expansive transformation ϕ (the inflation factor)
- a rule that allows one to divide each ϕT_i into copies of the $T_1, T_2, \ldots, T_m$

A tile-substitution s with expansion ϕ is a map $T_i \mapsto s(T_i)$, where $s(T_i)$ is a patch made of translates of the prototiles and

$$\phi(T_i) = \bigcup_{T_j \in s(Ti)} T_j$$

Example



Effectiveness for S-adic subshifts

- Directive sequences
- Pattern frequencies/invariant measure
- Language
- Existence of (decorated) local rules (being sofic or an SFT)

There are mainly two difficulties which come from

- the notion of substitution in dimension d
- the S-adic framework

A natural viewpoint since the characterization of entropy as right-recursively enumerable numbers [Hochman-Meyerovitch]

Existence of local rules

A closed subset $\mathbf{S} \subset \mathfrak{S}^{\mathbb{N}}$ is effectively closed if the set of (finite) words which do not appear as prefixes of elements of $\mathbf{S}$ is recursively enumerable.

One enumerates forbidden prefixes.

- Theorem [Aubrun-Sablik] We consider rectangular substitutions. The $\mathbf{S}$-adic subshift $X_{\mathbf{S}}$ is sofic if and only if it can be defined by a set of directive sequences $\mathbf{S}$ which is effectively closed.

  [Hochman 2009, Durand-Romashchenko-Shen 2010, Aubrun-Sablik 2013]

  Every effective subshift X can be simulated by a higher dimensional sofic subshift

  $X^{\mathbb{Z}}$ is the recoloring of a SFT

  1D effective $\leftrightarrow$ 2D SFT

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- A similar result for more general substitutions is expected
- The difficulty relies in the ability to exhibit a rectangular grid to use the simulation of a one-dimensional effective subshift by a two-dimensional sofic subshift

Effectiveness for S-adic shifts

Theorem [B.-Fernique-Sablik] Let X_S be a strictly ergodic S-adic subshift defined with respect to a directive sequence $S \in \mathfrak{S}^{\mathbb{N}}$ such that $\mathfrak{S}$ satisfies the good growing property. The following conditions are equivalent:

- there exists a computable sequence S' such that $X_S = X_{S'}$;
- the unique invariant measure of X_S is computable;
- the subshift X_S is decidable.

Good growing substitution

- A finite set of substitutions $\mathfrak{S}$ has a good growing property if
  - there are finitely many ways of gluing super-tiles: there exists a finite set of patterns $\mathcal{P} \subset \mathcal{A}^*$ such that if a pattern formed by a super-tile of order n surrounded by super-tiles of order n is in the language of $X_{\mathfrak{S}^{\mathbb{N}}}$, then it appears as the n-iteration of a pattern of $\mathcal{P}$
  - the size of the super-tiles of order n grows with n: for every ball of radius R, there exist $n \in \mathbb{N}$ such a translate of this ball is contained in all the supports of super-tiles of order n.
- Non-trivial rectangular substitutions or geometrical tiling substitutions verify this property.

Let $\mathbf{S} \subset \mathfrak{S}^{\mathbb{N}}$ be a closed subset. If $X_{\mathbf{S}}$ is effective, then there exists an effective closed subset $\mathbf{T} \subset \mathfrak{S}^{\mathbb{N}}$ such that $X_{\mathbf{S}} = X_{\mathbf{T}}$. The reciprocal is true if $\mathfrak{S}$ has the good growing property.

We assume that the unique invariant measure of X_S is computable.

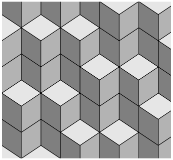
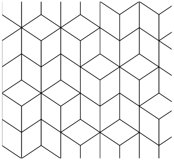
Let d stand for the cardinality of the alphabet of the substitutions in $\mathfrak{S}$. The letter frequency vector is in the cone defined by the product of the incidence matrices of the directive sequence. The letter frequency belongs to the cone

$$\bigcap_n M_1 \cdots M_n \mathbb{R}_+^d$$

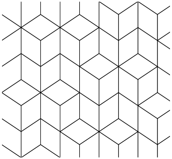
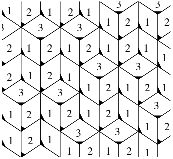
which is one-dimensional by unique ergodicity.

Given a precision e, one can compute n such that the columns of $M_1 \cdots M_n$ are expected to be at a distance less than e from the letter frequency vector. We fix a cylinder around the direction provided by the letter frequency vector with precision e. Now we test finite products of n substitutions in $\mathfrak{S}$. We consider the cone obtained by taking the product of the incidence matrices, and check whether it intersects the cylinder. If it does not intersect the cylinder, one gets a forbidden product of substitutions, which proves that $\{S\}$ is effectively closed.

2D Sturmian words

From a discrete plane to a tiling by projection....

....and from a tiling by lozenges to a ternary coding

Two-dimensional Sturmian words

Theorem [B.-Vuillon]

Let $(u_{\mathbf{m}})_{\mathbf{m}\in\mathbb{Z}^2} \in \{1,2,3\}^{\mathbb{Z}^2}$ be a 2d Sturmian word, that is, a coding of a discrete plane. Then there exist $x \in \mathbb{R}$, and α, $\beta \in \mathbb{R}$ such that $1, \alpha, \beta$ are $\mathbb{Q}$-linearly independent and $\alpha + \beta < 1$ such that

$$\forall \mathbf{m} = (m, n) \in \mathbb{Z}^2,\ U_{\mathbf{m}} = i \Longleftrightarrow R_\alpha^m R_\beta^n(x) = x + m\alpha + n\beta \in I_i \pmod 1,$$

with

$$I_1 = [0, \alpha[,\ I_2 = [\alpha, \alpha+\beta[,\ I_3 = [\alpha+\beta, 1[$$

or

$$I_1 =]0, \alpha],\ I_2 =]\alpha, \alpha+\beta],\ I_3 =]\alpha+\beta, 1].$$

Coding of a $\mathbb{Z}^2$-action

Effective $2d$ Sturmian shifts

Theorem [B.-Bourdon-Jolivet-Siegel] A $2d$ Sturmian shift is S-adic with an expansion provided by Brun continued fraction algorithm

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Theorem [B.-Fernique-Sablik] Let X be a $2d$ Sturmian shift. The following conditions are equivalent:

- its normal vector is computable;
- its unique invariant measure is computable;
- its language is decidable;
- Its Brun S-adic directive sequence is computable.

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Theorem [Fernique-Sablik] A Euclidean plane E admits colored weak local rules (there is a sofic shift that contains planar tilings with a slope parallel to E with a bounded thickness) if and only if it is computable.

Some decision problems for substitutions

Decision problems for word substitutions

Some classical decision problems for primitive substitutions can be solved using return words and derived sequences [Durand]

Let $\mathcal{A}$, $\mathcal{B}$, be finite alphabets. We consider two morphisms $\sigma\colon \mathcal{A}^* \to \mathcal{A}^*$, $\phi\colon \mathcal{A}^* \to \mathcal{B}^*$; an infinite word of the form

$$\lim_n \sigma^n(u)$$

is is a D0L word and

$$\phi(\lim_n \sigma^n(u))$$

an HD0L or morphic word, for u finite word.

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Let σ be a primitive substitution. It generates a minimal subshift X_σ. A return word to a word u of its language is a word w of the language such that

uw admits exactly two occurrences of u, with the second occurrence of u being a suffix of uw.

One can recode sequences of the subshift via return words, obtaining derived sequences.

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Some classical decision problems for primitive substitutions can be solved using return words and derived sequences [Durand]

- The HD0L ω-equivalence problem for primitive morphisms: it is decidable to know whether two HD0L words are equal.
- The decidability of the ultimate periodicity of HD0L infinite sequences: it is decidable to know whether an HD0L word is ultimately periodic.
- The uniform recurrence of morphic sequences is decidable.

Recognizability

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Let $\mathcal{A}$ be a finite alphabet, let $T : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ be the shift map.

Dynamic recognizability Let $\sigma : \mathcal{A} \to \mathcal{A}^*$ be a substitution and $y \in \mathcal{A}^{\mathbb{Z}}$.

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• If $y = T^k\sigma(x)$ with $x \in \mathcal{A}^{\mathbb{Z}}$, $k \in \mathbb{Z}$, then we say that (k, x) is a σ-representation of y.

• If $0 \leq k < |\sigma(x_0)|$, then (k, x) is a centered σ-representation of y.

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We say that σ is recognizable in X_σ if each aperiodic $y \in \mathcal{A}^{\mathbb{Z}}$ has at most one centered σ-representation in X_σ.

Recognizability for substitutions

Theorem [Mossé, 92 & 96] A primitive, aperiodic substitution σ is recognizable in X_σ.

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Theorem [Bezuglyi-Kwiatkowski-Medynets 2009] An aperiodic substitution σ is recognizable in X_σ.

Theorem [B.-Steiner-Thuswaldner-Yassawi] A substitution σ is recognizable in X_σ for aperiodic points.

Beyond substitutions: S-adic shifts

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For $0 \leq n < N$, let

$$\sigma_{[n,N)} = \sigma_n \circ \sigma_{n+1} \circ \cdots \circ \sigma_{N-1}.$$

For $n \geq 0$, the languages $\mathcal{L}_{\boldsymbol{\sigma}}^{(n)}$ associated with $\boldsymbol{\sigma}$ are defined by

$$\mathcal{L}_{\boldsymbol{\sigma}}^{(n)} = \left\{ w \in \mathcal{A}_n^* : w \text{ is a subword of } \sigma_{[n,N)}(a), \text{ some } a \in \mathcal{A}_N,\ N > n \right\}.$$

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Let $(X_{\boldsymbol{\sigma}}^{(n)}, T)$, the shift generated by $\mathcal{L}_{\boldsymbol{\sigma}}^{(n)}$.

Set $X_{\boldsymbol{\sigma}} = X_{\boldsymbol{\sigma}}^{(0)}$ and call $(X_{\boldsymbol{\sigma}}, T)$ the S-adic shift generated by the directive sequence $\boldsymbol{\sigma}$.

Recognizability for morphisms and S-adic shifts

- Given $X \subseteq \mathcal{A}^{\mathbb{Z}}$ and $\sigma : \mathcal{A} \to \mathcal{B}^+$, we say that σ is recognizable in X if each $y \in \mathcal{B}^{\mathbb{Z}}$ has at most one centered σ-representation in X.

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• A directive sequence $\boldsymbol{\sigma}$ is recognizable at level n if σ_n is recognizable in $X_{\boldsymbol{\sigma}}^{(n+1)}$.

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- The sequence $\boldsymbol{\sigma}$ is recognizable if it is recognizable at level n for each $n \geq 0$.

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- The sequence $\boldsymbol{\sigma}$ is recognizable if it is recognizable at level n for each $n \geq 0$.

- If there is an $n_0 \in \mathbb{N}$ such that $\boldsymbol{\sigma}$ is recognizable at level n for each $n \geq n_0$, then we say that $\boldsymbol{\sigma}$ is eventually recognizable.

Full recognizability for morphisms

Theorem [B.-Steiner-Thuswaldner-Yassawi] Let $\sigma : \mathcal{A} \to \mathcal{B}^+$ be a morphism such that

- $\mathrm{rk}(M_\sigma) = |\mathcal{A}|$, or
- $|\mathcal{A}| = 2$, or
- σ is (rotationally) a left or right permutative morphism.

Then σ is fully recognizable for aperiodic points (at most one centred representation in $\mathcal{A}^{\mathbb{Z}}$).

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Theorem [B.-Steiner-Thuswaldner-Yassawi] Let $\boldsymbol{\sigma} = (\sigma_n)_{n \geq 0}$ be a sequence of morphisms with $\sigma_n : \mathcal{A}_{n+1} \to \mathcal{A}_n^+$ with bounded alphabets. If each morphism σ_n satisfies one of

- $\mathrm{rk}(M_{\sigma_n}) = |\mathcal{A}_{n+1}|$, or
- $|\mathcal{A}_{n+1}| = 2$, or
- σ_n is (rotationally) a left or right permutative morphism,

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Theorem[B.-Steiner-Thuswaldner-Yassawi] Let $\boldsymbol{\sigma} = (\sigma_n)_{n\geq 0}$ be a sequence of morphisms with $\sigma_n : \mathcal{A}_{n+1} \to \mathcal{A}_n^+$ such that $\liminf_{n\to\infty} |\mathcal{A}_n| < \infty$ and $\sup_{n\geq 0} |\{\mathcal{L}_x : x \in X_{\boldsymbol{\sigma}}^{(n)}\}| < \infty$. Then $\boldsymbol{\sigma}$ is eventually recognizable for aperiodic points.

Conclusion and perspectives

The following effectiveness notions for S-adic symbolic dynamical systems are intimately related

- Effectiveness of the directive sequences
- Computability of pattern frequencies/invariant measures
- Decidability of the language
- Existence of (colored) local rules (being sofic or an SFT)

How to extend these results?

- Substitutive subshifts
- S-adic subshifts
  - Dendric subshifts
    - Sturmian subshifts
    - Arnoux-Rauzy subshifts
    - Interval exchanges