On expansion of numbers in bases 2 and 3

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Decidability and dynamical systems

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- We represent numbers using sequences of digits.
  Decimal (base 10): $0.2016 = \frac{2}{10} + \frac{0}{10^2} + \frac{1}{10^3} + \frac{6}{10^4}$
  Binary (base 2): $0.11001 = \frac{1}{2} + \frac{1}{2^2} + \frac{0}{2^3} + \frac{0}{2^4} + \frac{1}{2^5}$

- Problems: What do we know about the digit sequence of a given number (such as $\pi, e, \sqrt{2}, \ln(2)$)?
  (How the digit sequence of a number is related to its algebraic and analytical properties?)
- $\rightarrow$ very poor understanding!

A famous example

$$\begin{aligned}\pi &= 4\prod_{n=1}^{\infty}\frac{2n(2n+2)}{(2n+1)^2}\\ &= \frac{4}{1}-\frac{4}{3}+\frac{4}{5}-\frac{4}{7}+\frac{4}{9}-\frac{4}{11}+\frac{4}{13}+\cdots\\ &= \sum_{n=1}^{\infty}\frac{1}{16^n}\left(\frac{4}{8n+1}-\frac{2}{8n+4}-\frac{1}{8n+5}-\frac{1}{8n+6}\right)\\ \ldots &\quad \text{many other formulas}\end{aligned}$$

Question: What do we know about the digit sequence of π?

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$$\pi = 3.\ 1415926535897932384626433832$$
$$7950288419716939937510582 09749$$
$$445923078164062862089986280348$$
$$253421170679821480865132823066$$
$$4709384460955058223172535940\cdots$$

Numerical computations show that it behaves "randomly".
But we don't even know if the digit 0 appears infinitely often!

$\rightarrow$ Randomness?

Randomness of digit sequence

Randomness $\to$ Complexity of the digit sequence.

- For $x \in [0,1]$, we denote the orbit of x under $\times 10$ by

$$\mathcal{O}_{10}(x) = \{10^k x \mod 1 : k \in \mathbb{N}\}.$$

$C_{10}(x) := \dim_{\mathrm{H}} \overline{\mathcal{O}_{10}(x)} \to$ complexity of x in base 10.

- Easy fact: if $x \in \mathbb{Q} \Rightarrow C_m(x) = 0$; how about $x \notin \mathbb{Q}$?
- **We don't understand this!**

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- If $\log p / \log q \in \mathbb{Q} \Rightarrow C_p(x) = C_q(x)$. ex: $C_{10}(x) = C_{100}(x)$.
- And if $\log p / \log q \notin \mathbb{Q}$?

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- If $\log p / \log q \in \mathbb{Q} \Rightarrow C_p(x) = C_q(x)$. ex: $C_{10}(x) = C_{100}(x)$.
- And if $\log p / \log q \notin \mathbb{Q}$?
- **Furstenberg**: if $\log p / \log q \notin \mathbb{Q}$, then $\mathcal{O}_p(x)$ and $\mathcal{O}_q(x)$ can not both be "simple"!

Conjecture 1 (Furstenberg, 1969)

If $\log p / \log q \notin \mathbb{Q}$, then for all $x \in [0,1] \setminus \mathbb{Q}$, we have

$$\dim_{\mathrm{H}} \overline{\mathcal{O}_p(x)} + \dim_{\mathrm{H}} \overline{\mathcal{O}_q(x)} \geq 1.$$

$\Rightarrow$ Low complexity in one base implies high complexity in another base.

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- Applying Conjecture 1 to $\sqrt{2}, e, \pi, \ln 2, \ldots$

$$\textit{Example}: \quad \dim_{\mathrm{H}} \overline{\mathcal{O}_2(\pi)} + \dim_{\mathrm{H}} \overline{\mathcal{O}_{10}(\pi)} \geq 1 \quad !$$

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  instead of p-adic expansion $\to$ continued fraction.
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Problem: For $p \geq 2$ and **all** $x \in [0,1] \setminus \mathbb{Q}$, we have

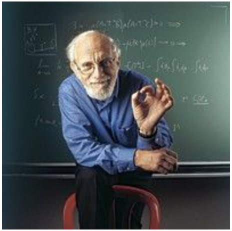
$$\dim_{\mathrm{H}} \overline{\mathcal{O}_G(x)} + \dim_{\mathrm{H}} \overline{\mathcal{O}_p(x)} \geq 1.$$

Here $\mathcal{O}_G(x)$ is the orbit of x under the Gauss map.

- Example: for $x = \sqrt{2}$, the $\mathcal{O}_G(x)$ is periodic, so $\dim_{\mathrm{H}} \overline{\mathcal{O}_G(x)} = 0$,

$$\textit{Problem} \Rightarrow \dim_{\mathrm{H}} \overline{\mathcal{O}_p(\sqrt{2})} = 1!$$

H. Furstenberg: a great mathematician with a remarkable originality.



- Ergodic theory and dynamical systems;
- Applying ergodic theory to combinatorics and number theory; (Szemerédi theorem, Green-Tao's work)
- $\times 2, \times 3$-rigidity problem in ergodic theory; (Lindenstrauss' work)
- Disjointness of dynamical systems
- Products of random matrices, non-commutative ergodic theory;
- Unique ergodicity of horocycle flow, toral maps, ...
- Analysis on symmetric spaces and homogeneous flows ...
- Wolf prize (2006).

Conjecture 1 (Furstenberg, 1969)

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Conjecture 1' (Furstenberg)

Suppose $\log p / \log q \notin \mathbb{Q}$. There exists a set $E \subset [0,1]$ with $\dim_{\mathrm{H}} E = 0$ such that for all $x \in [0,1] \setminus E$, we have

$$\dim_{\mathrm{H}} \overline{\mathcal{O}_p(x)} + \dim_{\mathrm{H}} \overline{\mathcal{O}_q(x)} \geq 1.$$

Note that we always have $\mathbb{Q} \subset E$.
Object.

From Conjecture 1 to intersections of Cantor sets.

- Recall $\mathcal{O}_p(x) = \{p^n x \mod 1 : n \in \mathbb{N}\}$.
- Let $A = \overline{\mathcal{O}_2(x)}$ and $B = \overline{\mathcal{O}_3(x)}$.
- Suppose $\dim_{\mathrm{H}} A + \dim_{\mathrm{H}} B < 1$, then Conjecture 1 $\Rightarrow x \in \mathbb{Q}$.
- And more generally, we have $A \cap B \subset \mathbb{Q}$.
- $\rightarrow$ the intersection $A \cap B$ is small.

- $\rightarrow$ reason?
- $\rightarrow$ not individual phenomena.
- $\rightarrow$ more general setting.

Problem of Furstenberg (1969): intersections of Cantor sets

- Dynamical system (X, f): X compact metric space, $f : X \to X$.
- Principal example : $X = [0,1]$, $f : x \mapsto 2x \mod 1$.
- f-invariant sets:

$$\{A \subset X : A \textit{ compact }, f(A) \subset A\}.$$

$\to$ display dynamical properties of f.

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- $(X, f), (X, g)$: compare the two systems $\to$ compare their invariant sets.
- Particularly: when f et g are "independent" $\to$ few common dynamical structures $\to$ the intersections of their invariant sets should be "as small as possible".
- independence: arithmetical or geometrical
- Ex: $f : x \to 2x \mod 1$, $f : x \to 3x \mod 1$ ($\log 2/\log 3 \notin \mathbb{Q} \to$ multiplicatively independent)

Transversality of dynamical systems (Furstenberg 1969)

- (X, f), (X, g); dim $\to$ a dimension function (ex. $\dim_{\mathrm{H}}$)
- Furstenberg: f et g are called transverse if for all $A = f$-invariant et $B = g$-invariant, we have

$$\dim A \cap B \leq \max\{0, \dim A + \dim B - \dim X\}$$

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Conjecture 2 (Furstenberg 1969)

Let $p, q \in \mathbb{N}_{\geq 2}$ with $\log p / \log q \notin \mathbb{Q}$. If $A \subset [0, 1]$ is $\times p$-invariant and $B \subset [0, 1]$ is $\times q$-invariant, then for all $u, v \in \mathbb{R}$, we have

$$\dim_{\mathrm{H}}(uA + v) \cap B \leq \max\{0, \dim_{\mathrm{H}} A + \dim_{\mathrm{H}} B - 1\}.$$

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Object of this talk.

The intersection conjecture of Furstenberg is true.

Theorem (W, Ann. Math. 2019)

Let $p, q \in \mathbb{N}_{\geq 2}$ with $\log p / \log q \notin \mathbb{Q}$. If $A =$ p-invariant and $B =$ q-invariant, then for all $u, v \in \mathbb{R}$,

$$\overline{\dim}_{\mathrm{B}}(f(A) \cap B) \leq \max\{0, \dim_{\mathrm{H}} A + \dim_{\mathrm{H}} B - 1\}$$

for non-degenerated differentiable $f : \mathbb{R} \to \mathbb{R}$.

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Remark: The intersection conjecture of Furstenberg has been simultaneously and independently proved by **P. Shmerkin** (Ann. Math. 2019) using completely different (additive combinatorial) methods.

Historic paper of Furstenberg (1967):

"Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation"

- Disjointness of dynamical systems.
- The celebrated $\times 2, \times 3$-rigidity result: if $A \subset [0,1]$ is simultaneously $\times 2$ and $\times 3$-invariant and $\sharp A = \infty$, then $A = [0,1]$.
- $\rightarrow$ if $x \in [0,1] \setminus \mathbb{Q}$, then $\{2^n 3^m x \mod 1\}_{n,m \in \mathbb{N}}$ is dense in $[0,1]$.
- The famous $\times 2, \times 3$ conjecture: if μ is a non-atomic measure, ergodic and simultaneously $\times 2$ and $\times 3$-invariant, then $\mu = \mathcal{L}$.
  (Partial solution: Rudolph; influenced: Einsiedler-Katok-Lindenstrauss work)

Problem 1967 $\rightarrow$ "intersections of invariant measures";
Problem 1969 $\rightarrow$ "intersections des invariant sets".

- Cobham's theorem/theory (in 1960's)

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- Fabien, ...

Known results about Conjecture 2 (intersection Conjecture)

- Observation: for $A, B \subset \mathbb{R}$, $u, v \in \mathbb{R}$, we have

$$(uA + v) \cap B \xrightarrow{\text{affine copy}} (A \times B) \cap \ell_{u,v},$$

where $\ell_{u,v} = \{(x, y) : y = ux + v\}$.

- Classical result on sections of fractals:

Theorem (classical)

Let $E \subset \mathbb{R}^2$ be a Borel set. Then for each $u \in \mathbb{R}$,

$$\dim_{\mathrm{H}}(E \cap \ell_{u,v}) \le \max\{0, \dim_{\mathrm{H}} E - 1\} \quad \text{for} \quad \mathcal{L}\text{-almost every} \quad v.$$

- Consequence: if $A = \times p\text{-inv}$ and $B = \times q\text{-inv}$, then for each u

$$\dim_{\mathrm{H}}(uA + v) \cap B \le \max\{0, \dim_{\mathrm{H}} A + \dim_{\mathrm{H}} B - 1\} \quad \text{for} \quad \mathcal{L}\text{-a.e.} \quad v$$

Proof: $E = A \times B$; $\dim_{\mathrm{H}} A \times B = \dim_{\mathrm{H}} A + \dim_{\mathrm{H}} B$.

- Furstenberg is true for "almost all" sections.

Partial results concerning all sections: Furstenberg

Theorem (Furstenberg, 1969)

Under the hypothesis of Conjecture 2, if there exist u_0, v_0 such that $\overline{\dim}_{\mathrm{B}}(u_0 A + v_0) \cap B = \gamma > 0$, then for $\mathcal{L}$-a.e. u, $\exists$ v such that

$$\dim_{\mathrm{H}}(uA + v) \cap B \geq \gamma.$$

As a (simple) corollary, for all u, v, we have

$$\overline{\dim}_{\mathrm{B}}(uA + v) \cap B \leq \max\{0, \dim_{\mathrm{H}} A + \dim_{\mathrm{H}} B - \frac{1}{2}\}.$$

Method (Furstenberg): CP-process/CP-chain.

Consequence: Conjecture 2 is true under the condition

$$\dim_{\mathrm{H}} A + \dim_{\mathrm{H}} B < \frac{1}{2}.$$

Recent results (Feng-Huang-Rao, 2013)

- Feng-Huang-Rao: Affine embeddings of self-similar sets.
- $A = p$-Cantor set if $\exists D \subset \{0, \cdots, p-1\}$ such that

$$A = \{\sum_{k\geq 1} p^{-k} x_k : x_k \in D\}.$$

Remark: A is a self-similar and $\times p$-invariant set.

Theorem (Feng-Huang-Rao, JMPA 2013)

Let $p, q \in \mathbb{N}_{\geq 2}$ with $\log p / \log q \notin \mathbb{Q}$. If $A = p$-Cantor and $B = q$-Cantor, then there is no affine embedding between A and B.

As a consequence, $\exists\ \delta = \delta(A, B) > 0$ (non-effective) such that

$$\dim_{\mathrm{H}}(uA + v) \cap B \leq \min\{\dim_{\mathrm{H}} A, \dim_{\mathrm{H}} B\} - \delta.$$

Remark: for the Conjecture of Furstenberg, one expects

$$\delta = 1 - \max\{\dim_{\mathrm{H}} A, \dim_{\mathrm{H}} B\}.$$

Related recent results

- Adamczewski-Bell's results on k-automatic and ℓ-automatic fractals (TAMS 2011);
- Boigelot, Brusten, Bruyère, Jodogne, Leroux, Wolper: results on k-recognizable and ℓ-recognizable fractals
- Charlier-Leroy-Rigo's (generalized) results to graph directed IFSs (Adv. Math. 2015);
- Chan-Hare (PAMS 2015);
- Feng-Wang (Adv. Math. 2009), ...

The intersection conjecture of Furstenberg is true.

Theorem (W)

Let $p, q \in \mathbb{N}_{\geq 2}$ with $\log p / \log q \notin \mathbb{Q}$. If $A = p$-invariant and $B = q$-invariant, then for all $u, v \in \mathbb{R}$,

$$\overline{\dim}_{\mathrm{B}}(f(A) \cap B) \leq \max\{0, \dim_{\mathrm{H}} A + \dim_{\mathrm{H}} B - 1\}$$

for non-degenerated differentiable $f : \mathbb{R} \to \mathbb{R}$.

We have a more general version of the theorem on the intersections of incommensurable homogeneous self-similar sets. ($\rightarrow$ arbitrary contraction ratios.)

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Our methods also allow us to prove the followingp:

- Given $\beta > 1$, T_β: β-transformation.
- For $\log \alpha / \log \beta \notin \mathbb{Q}$, if A is T_α-invariant compact and B is T_β-invariant compact, then

$$\dim_{\mathrm{B}} A \cap (uB + v) \leq \max\{0, \dim_{\mathrm{H}} A + \dim_{\mathrm{H}} B - 1\}.$$

- Given k, ℓ with $\log k / \log \ell \notin \mathbb{Q}$, A=k-automatic, B=ℓ-automatic.
- Non-compact case: A is the set of generic points of a $\times 2$-invariant ergodic measure, B is the set of generic points of a $\times 3$-invariant ergodic measure.

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- Non-compact case: A is the set of generic points of a $\times 2$-invariant ergodic measure, B is the set of generic points of a $\times 3$-invariant ergodic measure. Another proof of Rudolph's theorem.
-

$$A = \{x \in \mathbb{T} : \frac{1}{n}\sum_{k=1}^{n} f(T_2^k x) \to a\}$$

$$B = \{x \in \mathbb{T} : \frac{1}{n}\sum_{k=1}^{n} g(T_3^k x) \to b\}$$

Optimality of Furstenberg's conjecture.

- Given $A = \times 2$-inv and $B = \times 3$-inv, for any $u \in \mathbb{R} \setminus \{0\}$, there are many v such that $\dim_{\mathrm{H}} A \cap (uB + v) = \max(0, \dim_{\mathrm{H}} A + \dim_{\mathrm{H}} B - 1)$.

"Theorem" (W, last week 2019)

Given A, B as above, for any $u \in \mathbb{R} \setminus \{0\}$, there is a set $V \subset \mathbb{R}$ with $\dim_{\mathrm{H}} V = 1$ such that for $v \in V$ we have

$$\dim_{\mathrm{H}} A \cap (uB + v) = \max(0, \dim_{\mathrm{H}} A + \dim_{\mathrm{H}} B - 1).$$

The problem we began with ...

Corollary (W)

Suppose $\log p / \log q \notin \mathbb{Q}$. There exists a set $E \subset [0,1]$ with $\dim_{\mathrm{H}} E = 0$ $(\dim_{\mathrm{P}} E = 0)$ such that for all $x \in [0,1] \setminus E$, we have

$$\dim_{\mathrm{H}} \overline{\mathcal{O}_p(x)} + \dim_{\mathrm{H}} \overline{\mathcal{O}_q(x)} \geq 1.$$

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- Bugeaud-Kim (Ann. Inst. Fourier 2017): for $x \notin \mathbb{Q}$, the expansions of x in base 2 and in base 3 cannot be both Sturmian.
- Adamczewski-Faverjon Conjecture (2018): let $\log k/\log \ell \notin \mathbb{Q}$, prove that a real number is automatic in both bases k and ℓ if and only if it is a rational number.
- Many open questions!!!

Thank you!