Aperiodic configurations in $\mathbb{Z}^2$-subshifts

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Part I

Subshift, domino, periodicity

- $\mathcal{A}$ a finite set called **alphabet** ($\{\square, \blacksquare\}$);
- $\mathbb{Z}^d$ a d-dimensional grid;
- $\mathcal{A}^*$ the set of (finite) **patterns**;
- $\mathcal{A}^{\mathbb{Z}^d}$ the set of (infinite) **configurations**.

$\mathcal{A}$ a finite set called **alphabet** ($\{\square, \blacksquare\}$);

$\mathbb{Z}^d$ a d-dimensional grid;

$\mathcal{A}^*$ the set of (finite) **patterns**;

$\mathcal{A}^{\mathbb{Z}^d}$ the set of (infinite) **configurations**.

Subshift: the set of configurations that do not contain any pattern from a given set of **forbidden patterns** $\mathcal{F} \subset \mathcal{A}^*$.

of finite type if $\mathcal{F}$ can be chosen finite.

A pattern is **locally admissible** if it does not contain any forbidden pattern.

Domino problem

Examples

$d = 1, \mathcal{A} = \{\square, \blacksquare\}, \mathcal{F} = \{\blacksquare\blacksquare\}$:

Domino problem

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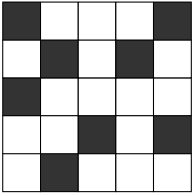
$d = 2, \mathcal{A} = \{\square, \blacksquare\}, \mathcal{F} = \{\blacksquare\blacksquare; \begin{smallmatrix}\blacksquare\\ \blacksquare\end{smallmatrix}\}$:



Domino problem

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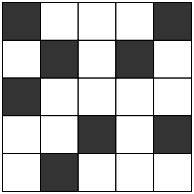

Domino problem

Input A subshift of finite type (a finite alphabet & finite set of forbidden patterns)

Output Does it contain a configuration?

Let's try in dimension 1

$\mathcal{A} = \{\square\blacksquare\}$,

$\mathcal{F} = \{\square\square\square\square, \square\square\blacksquare\square, \square\square\blacksquare\blacksquare, \square\blacksquare\square\blacksquare, \blacksquare\square\blacksquare\blacksquare, \blacksquare\blacksquare\blacksquare\blacksquare\}$

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No locally admissible pattern larger than $\square\blacksquare\blacksquare\blacksquare\square\blacksquare\square\square\square\blacksquare$.

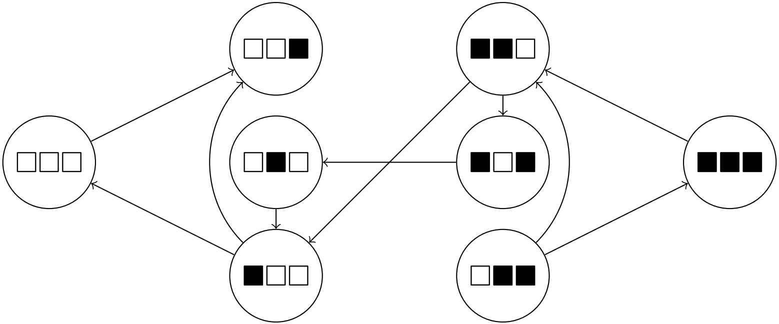
output: No

Let’s try in dimension 1

$\mathcal{A} = \{$□■$\}$,
$\mathcal{F} = \{$□□□□, □□■□, □□■■, □■□■, ■□■■, ■■■■$\}$

No locally admissible pattern larger than □■■■□■□□□■.

output: No



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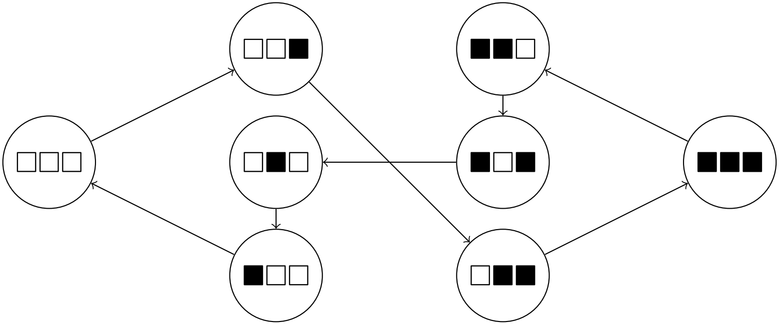
$\{$□□□□, □□■□, ■□□■, □■□■, ■■□□, ■□■■, □■■□, ■■■■$\}$

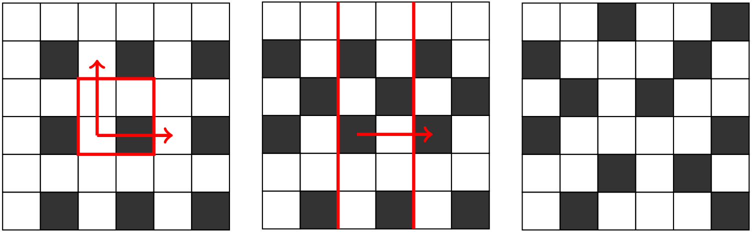
A periodic configuration:
. . . □□□■■■□■□■□□□■■■□■□■ . . . .

output: Yes



A configuration x **has period** $\vec{v} \in \mathbb{Z}^2$ if, for all coordinate $\vec{i} \in \mathbb{Z}^2$,

$$x_{\vec{i}+\vec{v}} = x_{\vec{i}}.$$



2 1 0

A configuration is k**-periodic** if it has k independent periods (and no more).

that is, the set of its periods has dimension k.

aperiodic := 0-periodic

Algorithm for the domino problem?

Domino problem

Input A subshift of finite type;

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Input A subshift of finite type;

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Algorithm sketch (Wang 1961)

For n ranging from 1 to ∞:

- Enumerate all locally admissible patterns $n \times n$
- If there is none: output **No**
- If there is a 2-periodic one: output **Yes.**
- Otherwise: continue.

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Algorithm sketch (Wang 1961)

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Negative part

If a subshift is empty, then it admits no locally admissible $n \times n$ pattern for some n.

(the problem is **co-semi-decidable**).

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Positive part

Does all nonempty subshifts have a 2**-periodic configuration** ?

Proposition

If a subshift contains a 1-periodic configuration, then it contains a 2-periodic one.

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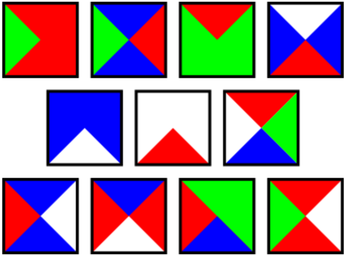
therefore, the previous algorithm does not halt.

Jeandel-Rao tiles:



Corollary

The domino problem is **co-semi-decidable-complete** (in dimension ≥ 2).

Proof

Use an aperiodic subshift to embed the computation of a Turing machine, so that the subshift is empty iff the machine halts.

Part II

Aperiodic domino problem

Periodic domino problem

Input An SFT ;

Output Does it contain a 2-periodic configuration?

semi-decidable-complete

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Aperiodic domino problem

Input An SFT ;

Output Does it contain an aperiodic configuration?

co-semi-decidable-hard

(all decidable in dimension 1)

Aperiodic domino problem

Input An SFT ;

Output Does it contain an aperiodic configuration?

Some thoughts

- How could we check that no configuration manages to avoid all periods?

A pattern / configuration x **avoids** period $\vec{p}$ in $\vec{i}$ iff $x_{\vec{i}} \neq x_{\vec{i}+\vec{p}}$.

Aperiodic domino problem

Input An SFT ;

Output Does it contain an aperiodic configuration?

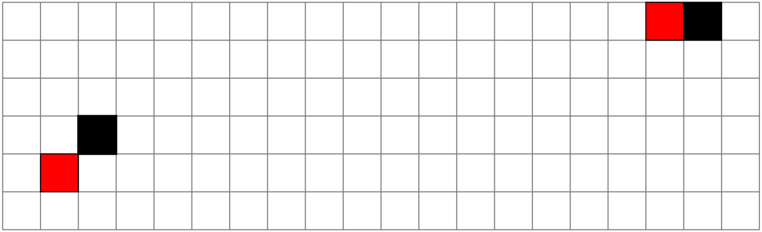
Some thoughts

- How could we check that no configuration manages to avoid all periods?
- Enumerate all locally admissible $n \times n$ patterns;
- if no pattern avoids all “small” periods, what can we conclude?

A pattern / configuration x **avoids** period $\vec{p}$ in $\vec{i}$ iff $x_{\vec{i}} \neq x_{\vec{i}+\vec{p}}$.

Assume that x avoids both periods $(1, 0)$, $(1, 1)$.

We demonstrate the key argument by gathering two avoidance points.



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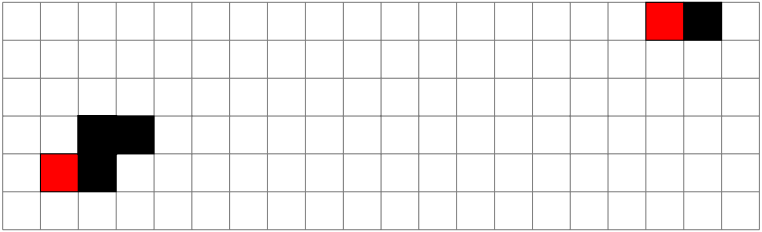

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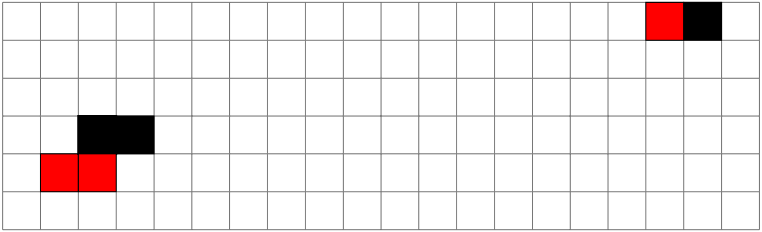

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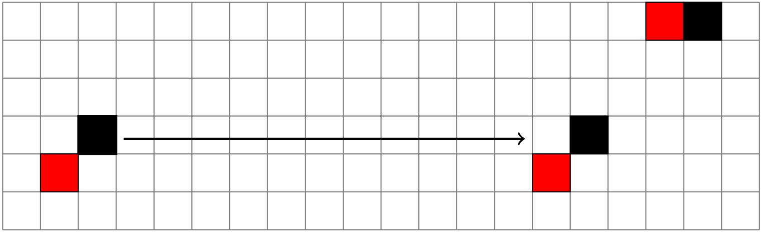

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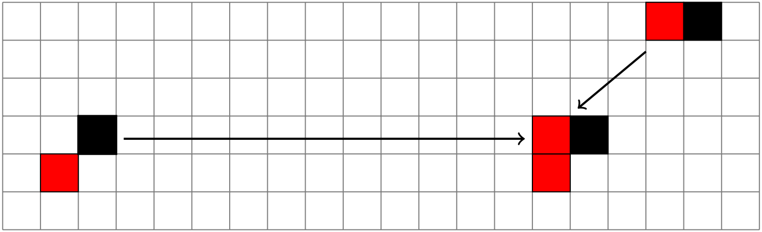
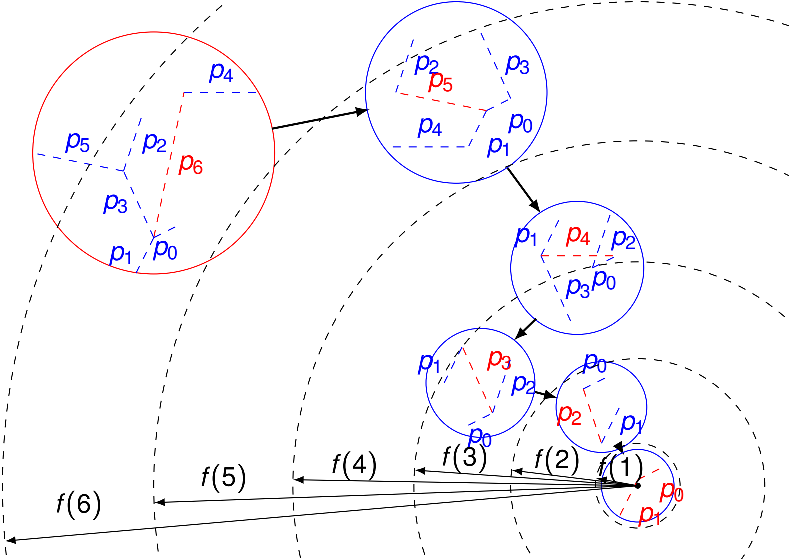


Shepherding avoidance points



Shepherding lemma

Assume that a configuration x avoids all periods $\vec{p} \in \mathcal{P}$.

Then there is a configuration in $\overline{\mathcal{O}(x)}$ that avoids all periods $\vec{p} \in \mathcal{P}$ with $\|\vec{p}\| \leq n$ in a central ball of radius $f(n)$ pour tout n,

where f is a universal computable fonction.

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Corollary (Grandjean, H., Vanier)

The aperiodic domino problem

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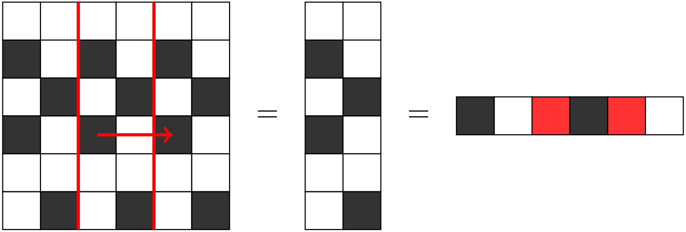
Algorithm

- Enumerate all locally admissible $n \times n$ patterns.
- If no pattern avoids all periods $\vec{p}$ s.t. $\|\vec{p}\| \leq k$ in a central ball $f(k) \times f(k)$: output **No**.
- Otherwise: continue.

Part III

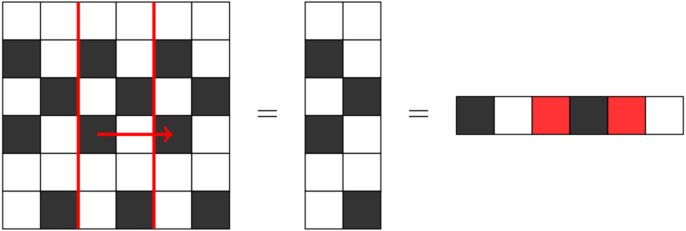
Subshifts with no aperiodic configurations



Proposition

In a $\mathbb{Z}^2$-subshift, the subspace of configurations with a given period is conjugate to a $\mathbb{Z}$-subshift.

Subshifts with no aperiodic configuration



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Théorème (Grandjean, H., Vanier)

Let X be a subshift with no aperiodic configuration.

There is a finite set of periods $\{\vec{p_1}, \ldots, \vec{p_n}\}$ such that any configuration in X has some period $\vec{p_i}$.

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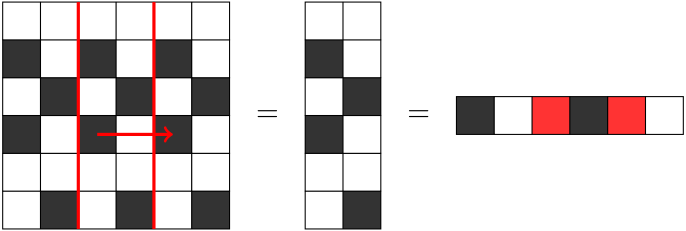

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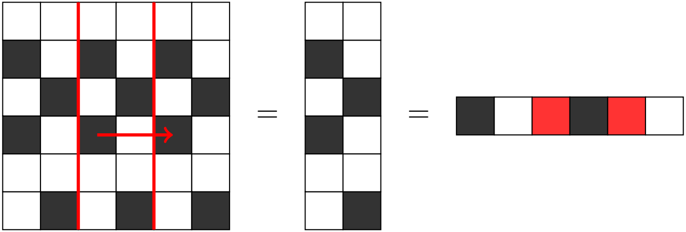
Proof

Previous result + compactness

Subshifts with no aperiodic configuration



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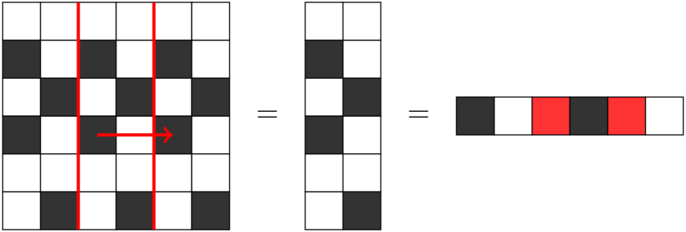
Corollary (Grandjean, H., Vanier)

Any $\mathbb{Z}^2$-subshift with no aperiodic configuration is **almost conjugate** to a $\mathbb{Z}$-subshift.



Corollary (Grandjean, H., Vanier)

Any $\mathbb{Z}^2$-subshift with no aperiodic configuration is **almost conjugate** to a $\mathbb{Z}$-subshift.

Proof

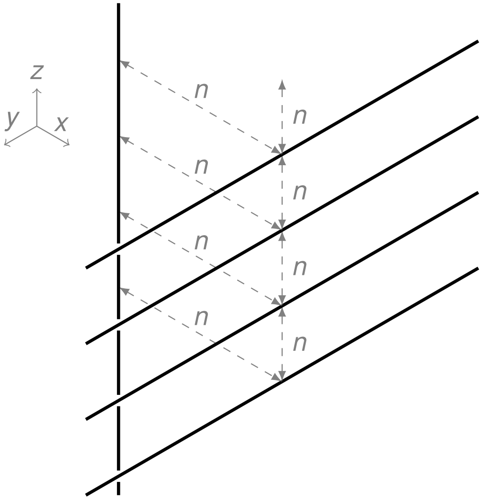
Choose your direction outside the finite set of periods.

Conclusion

- If some configuration avoids some set of periods, we can bound how far apart it does so and this makes compacity arguments work;
- For $\mathbb{Z}^2$-finite type subshifts, this makes the aperiodic domino problem as simple as we could hope.
- Some structural results in the non-finite type case.

Conclusion

- If some configuration avoids some set of periods, we can bound how far apart it does so and this makes compacity arguments work;
- For $\mathbb{Z}^2$-finite type subshifts, this makes the aperiodic domino problem as simple as we could hope.
- Some structural results in the non-finite type case.
- What about $\mathbb{Z}^d$, $d \geq 3$?
  - Lemmas are false (two lines need not intersect)
  - Results are false for general subshifts
  - How can we use the finite type hypothesis?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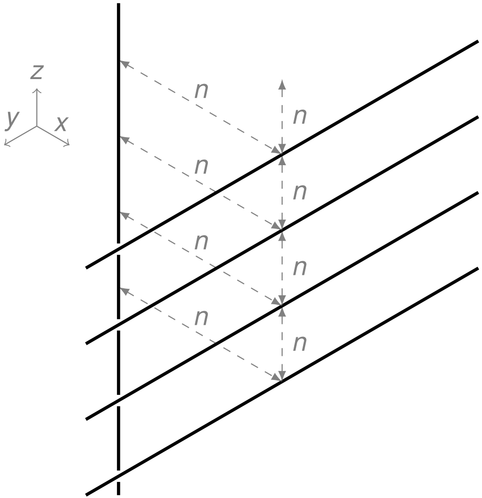
Conjectures (work in progress)

- We can gather avoidance points around a subspace of dimension $d-2$.
- The finite type hypothesis might let us gather around a point.



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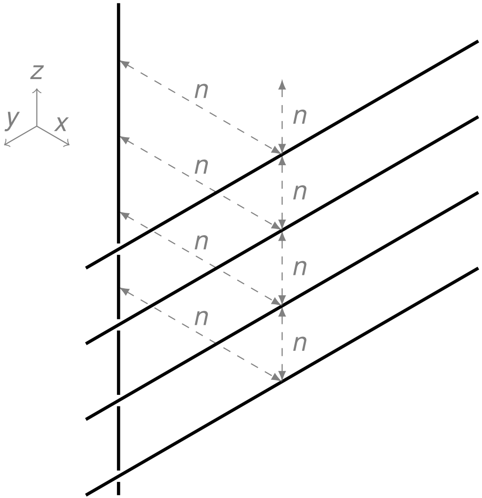

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